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LETTER TO THE EDITOR

Scattering amplitudes for supersymmetric shape-invariant potentials by operator methods

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Abstract. The scattering amplitudes for all currently known supersymmetric shape-invariant potentials are calculated by analytically continuing the explicit wavefunctions obtained recently via supersymmetric operator techniques. The procedure is illustrated in detail for the superpotential $W(x) = A \tanh \alpha x + B \operatorname{sech} \alpha x$, for which the S matrix has not been previously calculated.

Recently, the familiar harmonic oscillator raising and lowering operator technique has been generalised to other potentials of physical interest (Gendenshtein 1983, Dutt *et al* 1986, 1988). In particular, using the operator technique, bound-state eigenvalues and eigenfunctions have been obtained for all known shape-invariant potentials. The purpose of this letter is to obtain analytic expressions for the scattering amplitudes for shape-invariant potentials. Our approach is very general and is based on the explicit wavefunctions obtained recently (Dabrowska *et al* 1988) via supersymmetric operator techniques. Scattering amplitudes are calculated by analytically continuing the wavefunctions so that boundary conditions appropriate to scattering are satisfied. An alternative method was suggested very recently by Cooper *et al* (1987). It is rather elegant but, as we shall discuss below, it suffers from very restricted applicability. We shall illustrate both approaches by explicitly working out the reflection and transmission coefficients for the class of potentials

$$V(x) = A^2 + (B^2 - A^2 - A\alpha) \operatorname{sech}^2 \alpha x + B(2A + \alpha) \operatorname{sech} \alpha x \tanh \alpha x \qquad (1)$$

which results from the superpotential $W(x) = A \tanh \alpha x + B \operatorname{sech} \alpha x$. As far as we are aware, this exactly solvable potential has essentially been overlooked in the literature. The results for the scattering amplitudes of all currently known shape-invariant potentials are summarised in tables 1 and 2.

In supersymmetric quantum mechanics (Witten 1981, Cooper and Freedman 1983) the superpotential $W(x)$ determines the two-partner potentials $V_\pm(x) = W^2(x) \pm \mathrm{d}W/\mathrm{d}x$ ($\hbar = 2m = 1$). When supersymmetry is unbroken, the eigenstates of $V_\pm$ are related by

$$E_{n+1}^{(-)} = E_n^{(+)} \qquad E_0^{(-)} = 0 \qquad \psi_n^{(+)} \propto A\psi_{n+1}^{(-)} \qquad \psi_{n+1}^{(-)} \propto A^+\psi_n^{(+)} \qquad (2)$$

where

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad A^+ = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad W(x) = -\frac{\mathrm{d}}{\mathrm{d}x}[\ln \psi_0^{(-)}(x)]. \qquad (3)$$

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Table 1. Scattering amplitudes for all known shape-invariant one-dimensional potentials ($\hbar = 2m = 1$). The wavenumbers k and k' are defined by (5) in the text with $W_\pm = \lim_{x\to\pm\infty} W(x)$. Note that oscillator and infinite square well potentials, which are shape invariant, have no scattering and hence are not tabulated here.

Name of potential	Superpotential $W(x)$	Potential $V_-(x; a_0)$	Scattering amplitudes
Morse	$A - B\,\mathrm{e}^{-\alpha x}$	$A^2 + B^2\,\mathrm{e}^{-2\alpha x} - 2B\left(A + \frac{\alpha}{2}\right)\mathrm{e}^{-\alpha x}$	$R(k') = \frac{(2\lambda)^{-2\mathrm{i}q}\Gamma(-s-\mathrm{i}q)\Gamma(2\mathrm{i}q)}{\Gamma(-s+\mathrm{i}q)\Gamma(-2\mathrm{i}q)}$; $s \equiv \frac{A}{\alpha}$, $\lambda \equiv \frac{B}{\alpha}$, $q \equiv \frac{k'}{\alpha}$
Symmetric Rosen–Morse	$A\tanh\alpha x$	$A^2 - A(A+\alpha)\,\mathrm{sech}^2\,\alpha x$	$T(k) = \frac{\Gamma(-s-\mathrm{i}q)\Gamma(s+1-\mathrm{i}q)}{\Gamma(-\mathrm{i}q)\Gamma(1-\mathrm{i}q)}$; $R(k) = T(k)\frac{\mathrm{i}\sin\pi s}{\sinh\pi q}$; $s \equiv \frac{A}{\alpha}$, $q \equiv \frac{k}{\alpha}$
Rosen–Morse	$A\tanh\alpha x + \frac{B}{A}$	$A^2 + \frac{B^2}{A^2} + 2B\tanh\alpha x$ $-A(A+\alpha)\,\mathrm{sech}^2\,\alpha x$	$W_-^2 < E < W_+^2$: $R(k) = \frac{\Gamma(2\mathrm{i}q)\Gamma(-s+q'-\mathrm{i}q)\Gamma(s+1+q'-\mathrm{i}q)}{\Gamma(-2\mathrm{i}q)\Gamma(-s+q'+\mathrm{i}q)\Gamma(s+1+q'+\mathrm{i}q)}$ $E > W_+^2$: $T(k) = \frac{\Gamma(-s-\mathrm{i}q'-\mathrm{i}q)\Gamma(s+1-\mathrm{i}q'-\mathrm{i}q)}{\Gamma(-2\mathrm{i}q)\Gamma(1-2\mathrm{i}q')}$; $R(k) = \frac{T(k)\Gamma(2\mathrm{i}q)\Gamma(1-2\mathrm{i}q')}{\Gamma(-s-\mathrm{i}q'+\mathrm{i}q)\Gamma(s+1-\mathrm{i}q'+\mathrm{i}q)}$ $s \equiv \frac{A}{\alpha}$, $q \equiv \frac{k}{2\alpha}$, $q' \equiv \frac{k'}{2\alpha}$
	$A\tanh\alpha x + B\,\mathrm{sech}\,\alpha x$	$A^2 + (B^2 - A^2 - A\alpha)\,\mathrm{sech}^2\,\alpha x$ $+B(2A+\alpha)\,\mathrm{sech}\,\alpha x\tanh\alpha x$	$T(k) = \frac{\Gamma(-s-\mathrm{i}q)\Gamma(s+1-\mathrm{i}q)\Gamma(\frac{1}{2}+\mathrm{i}\lambda-\mathrm{i}q)\Gamma(\frac{1}{2}-\mathrm{i}\lambda-\mathrm{i}q)}{\Gamma(-\mathrm{i}q)\Gamma(1-\mathrm{i}q)\Gamma^2(\frac{1}{2}-\mathrm{i}q)}$ $R(k) = T(k)(\cos\pi s\sinh\pi\lambda\,\mathrm{sech}\,\pi q + \mathrm{i}\sin\pi s\cosh\pi\lambda\,\mathrm{cosech}\,\pi q)$ $s \equiv \frac{A}{\alpha}$, $\lambda \equiv \frac{B}{\alpha}$, $q \equiv \frac{k}{\alpha}$

Table 2. Scattering amplitudes for all known shape-invariant potentials in three dimensions. The wavenumber k' is defined by (5) in the text with $W_+ = \lim_{r\to\infty} W(r)$.

Name of potential	Superpotential $W(r)$	Potential $V_-(r; a_0)$	Scattering amplitudes
Coulomb	$\frac{e^2}{2(l+1)} - \frac{(l+1)}{r}$	$-\frac{e^2}{r} + \frac{l(l+1)}{r^2} - \frac{e^4}{4(l+1)^2}$	$S_l(k') = \frac{\Gamma(l+1-\mathrm{i}\varepsilon)\Gamma(1+\mathrm{i}\varepsilon)}{\Gamma(l+1+\mathrm{i}\varepsilon)\Gamma(1-\mathrm{i}\varepsilon)}; \quad \varepsilon \equiv \frac{e^2}{2k'}$
	$A \coth \alpha r - B \operatorname{cosech} \alpha r$	$A^2 + (B^2 + A^2 + A\alpha) \operatorname{cosech}^2 \alpha r - B(2A+\alpha) \coth \alpha r \operatorname{cosech} \alpha r$	$S_{l=0}(k') = \frac{2^{-4\mathrm{i}q}\Gamma(2\mathrm{i}q)\Gamma(-s-\mathrm{i}q)\Gamma(\lambda+\frac{1}{2}-\mathrm{i}q)}{\Gamma(-2\mathrm{i}q)\Gamma(-s+\mathrm{i}q)\Gamma(\lambda+\frac{1}{2}+\mathrm{i}q)}; \quad s \equiv \frac{A}{\alpha}, \quad \lambda \equiv \frac{B}{\alpha}, \quad q \equiv \frac{k'}{\alpha}$
Eckart	$-A \coth \alpha r + \frac{B}{A}$	$A^2 + \frac{B^2}{A^2} - 2B \coth \alpha r + A(A-\alpha) \operatorname{cosech}^2 \alpha r$	$S_{l=0}(k') = \frac{\Gamma(-2\mathrm{i}q)\Gamma(s+\mathrm{i}Q+\mathrm{i}q)\Gamma(s-\mathrm{i}Q+\mathrm{i}q)}{\Gamma(2\mathrm{i}q)\Gamma(s+\mathrm{i}Q-\mathrm{i}q)\Gamma(s-\mathrm{i}Q-\mathrm{i}q)};$ $s \equiv \frac{A}{\alpha}, \quad q \equiv \frac{k'}{2\alpha} = \frac{\{[A-(B/A)]^2 - E\}^{1/2}}{2\alpha},$ $Q \equiv \frac{k}{2\alpha} = \frac{\{[A+(B/A)]^2 - E\}^{1/2}}{2\alpha}$
Pöschl–Teller II	$A \tanh \alpha r - B \coth \alpha r$	$(A-B)^2 - A(A+\alpha) \operatorname{sech}^2 \alpha r + B(B-\alpha) \operatorname{cosech}^2 \alpha r$	$S_{l=0}(k') = \frac{\Gamma(2\mathrm{i}q)\Gamma(-s+\lambda-\mathrm{i}q)\Gamma(s+\lambda+\frac{1}{2}-\mathrm{i}q)}{\Gamma(-2\mathrm{i}q)\Gamma(-s+\lambda+\mathrm{i}q)\Gamma(s+\lambda+\frac{1}{2}+\mathrm{i}q)}; \quad s \equiv \frac{A}{2\alpha}, \quad \lambda \equiv \frac{B}{2\alpha}, \quad q \equiv \frac{k'}{2\alpha}$

Now, in order to have scattering, the potentials $V_\pm$ must be finite either at $x=-\infty$ or at $x=+\infty$ or both. Let $W(x=\pm\infty)=W_\pm$. Then for finite $W_\pm$, both $V_\pm$ have values $W_\pm^2$ at $x\to\pm\infty$.

Let us consider an incident plane wave e^{ikx} of energy E coming from $x\to-\infty$. As a result of scattering from the potentials $V_\pm$, one obtains transmitted waves $T_\pm(k)\,e^{ik'x}$ and reflected waves $R_\pm(k)\,e^{-ikx}$. In view of (2), one finds that the transmission and reflection amplitudes for the partner potentials $V_\pm$ are related by

$$R_-(k)=\left(\frac{W_-+ik}{W_--ik}\right)R_+(k) \qquad T_-(k)=\left(\frac{W_+-ik'}{W_--ik}\right)T_+(k) \tag{4}$$

where k and k' are given by

$$k=(E-W_-^2)^{1/2} \qquad k'=(E-W_+^2)^{1/2}. \tag{5}$$

Equation (4) is a fairly straightforward generalisation of a simpler situation when $W_+^2=W_-^2$ which was discussed by various people (Sukumar 1986, Akhoury and Comtet 1984). A few remarks are in order. (i) Clearly $|R_-|^2=|R_+|^2$ and $|T_-|^2=|T_+|^2$, i.e. the partner potentials have identical reflection and transmission probabilities. (ii) $R_-(T_-)$ and $R_+(T_+)$ have the same poles in the complex k plane except that $R_-(T_-)$ has an extra pole at $k=-iW_-$. This pole is on the positive imaginary axis only if $W_-<0$ (i.e. if SUSY is unbroken) in which case it corresponds to a zero-energy bound state. (iii) In the special case of $W_+=W_-$ we have $T_-(k)=T_+(k)$ while if $W_-=0$ then $R_-(k)=a-R_+(k)$. (iv) For the case of three-dimensional central potentials we analogously find that the scattering matrix $S(k')$ is given by $S_-(k')=[(W_+-ik')/(W_++ik')]S_+(k')$ where $W_+=W(r\to+\infty)$, and $S(k')=e^{i\delta(k')}$ where $\delta(k')$ is the phase shift.

The two supersymmetric partner potentials $V_\pm(x)$ are said to be shape invariant if

$$V_\pm(x,a_0)=V_-(x,a_1)+C(a_1) \tag{6}$$

where a_0 is a set of parameters and $a_1=f(a_0)$ is a function of a_0. Using (2), (3) and (6) Gendenshtein (1983) and Dutt *et al* (1986) have shown that shape-invariant potentials are analytically solvable, i.e.

$$E_n^{(-)}=\sum_{k=1}^{n}C(a_k) \qquad E_0^{(-)}=0 \qquad a_k=f^k(a_0) \tag{7}$$

$$\psi_n^{(-)}(x,a_0)\propto A^+(x,a_0)A^+(x,a_1)\ldots A^+(x,a_{n-1})\psi_0^{(-)}(x,a_n). \tag{8}$$

For shape-invariant potentials, the relation between $R_\pm$ and $T_\pm$ takes a particularly simple form (Cooper *et al* 1987):

$$R_-(k,a_0)=\left(\frac{W_-(a_0)+ik}{W_-(a_0)-ik}\right)R_-(k,a_1) \qquad T_-(k,a_0)=\left(\frac{W_+(a_0)-ik'}{W_-(a_0)-ik}\right)T_-(k,a_1). \tag{9}$$

Repeating the above step N times gives

$$R_-(k,a_0)=\prod_{n=0}^{N-1}\left(\frac{W_-(a_n)+ik}{W_-(a_n)-ik}\right)R_-(k,a_N) \tag{10}$$

$$T_-(k,a_0)=\prod_{n=0}^{N-1}\left(\frac{W_+(a_n)-ik'}{W_-(a_n)-ik}\right)T_-(k,a_N). \tag{11}$$

For all known shape-invariant potentials it turns out that $a_N=a_0\pm N\alpha$. Hence $T_-(k,a_0)(R_-(k,a_0))$ is known for all values of a_0 provided it is known in some strip, say $0 \leqslant a_0 < \alpha$. Unfortunately in most cases it is not that easy to know T or R for a finite range of parameters a_0. Hence this elegant approach has rather limited applicability.

We first present a straightforward approach for calculating scattering amplitudes by making use of (8), the operator expression for the bound-state wavefunctions. Using this expression, we have recently been able to obtain nth-state wavefunctions for all the known shape-invariant potentials (see table 1 of Dabrowska *et al* (1988)). Now in order to impose boundary conditions appropriate to the scattering problem, two modifications of the bound-state wavefunctions have to be made. (i) Instead of the parameter n labelling the number of nodes, one must use the wavenumber k' as given by (5) so that the asymptotic behaviour $\exp[-x(W_+^2 - E_n)^{1/2}]$ as $x \to +\infty$ is replaced by $e^{ik'x}$. (ii) The second solution of the Schrödinger equation must be kept—it has been discarded for bound-state problems since it diverged asymptotically.

To clarify the whole procedure, we now explicitly compute T and R for the class of potentials given by (1). The energy eigenstates are (Dabrowska *et al* 1988)

$$E_n = \alpha^2[S^2 - (S-n)^2] \tag{12a}$$

$$\psi_n = (1+y^2)^{-S/2} \exp(-\lambda \tan^{-1} y) \frac{\Gamma(n + i\lambda - S + \frac{1}{2})}{n!\Gamma(i\lambda - S + \frac{1}{2})} \times F\left(-n, n-2S, i\lambda - S + \tfrac{1}{2}; \frac{1+iy}{2}\right) \tag{12b}$$

where $S = A/\alpha$, $\lambda = B/\alpha$, $y = \sinh \alpha x$. On replacing n by $S + ik/\alpha$ we then obtain the following two independent scattering solutions:

$$F_1 = (1+y^2)^{S/2} \exp(-\lambda \tan^{-1} y) F\left(-S - \frac{ik}{\alpha}, -S + \frac{ik}{\alpha}, i\lambda - S + \tfrac{1}{2}, \frac{1+iy}{2}\right) \tag{13a}$$

$$F_2 = (1+y^2)^{-S/2} \exp(-\lambda \tan^{-1} y) \times \left(\frac{1+iy}{2}\right)^{1/2+S-i\lambda} F\left(\tfrac{1}{2} - i\lambda - \frac{ik}{\alpha}, \tfrac{1}{2} - i\lambda + \frac{ik}{\alpha}, \tfrac{3}{2} + S - i\lambda; \frac{1+iy}{2}\right). \tag{13b}$$

Now it is well known that if the asymptotic behaviour of $F_{1,2}$ is given by

$$\lim_{x \to \pm\infty} F_i(x) = a_{i\pm} e^{ikx} + b_{i\pm} e^{-ikx} \qquad (i = 1, 2) \tag{14}$$

then (for example, see ter Haar 1975)

$$T = \frac{a_{2+}b_{1+} - a_{1+}b_{2+}}{a_{2-}b_{1+} - a_{1-}b_{2+}} \qquad R = \frac{b_{1+}b_{2-} - b_{1-}b_{2+}}{a_{2-}b_{1+} - a_{1-}b_{2+}}. \tag{15}$$

By using standard identities of hypergeometric functions we can now extract the asymptotic behaviour of F_1 and F_2 and hence the coefficients $a_{i\pm}$, $b_{i\pm}$. We find

$$a_{1+} = D_1 \exp\left[\frac{\pi}{2}\left(\frac{k}{\alpha} - \lambda - iS\right)\right]_2 - S - (2ik/\alpha) \qquad b_{1-} = \exp\left[\left(-\frac{k}{\alpha} + \lambda + iS\right)\right]_{a_{1+}} \tag{16a}$$

$$b_{1+} = C_1 \exp\left[\frac{\pi}{2}\left(-\frac{k}{\alpha} - \lambda - iS\right)\right]_2 - S + (2ik/\alpha) \qquad a_{1-} = \exp\left[\pi\left(\frac{k}{\alpha} + \lambda + iS\right)\right]_{b_{1+}} \tag{16b}$$

$$a_{2+}=\mathrm{i}D_2\exp\left[\frac{\pi}{2}\left(-\frac{k}{\alpha}+\lambda+\mathrm{i}S\right)\right]_2 - S-(2\mathrm{i}k/\alpha) \qquad b_{2-}=-\exp\left[\pi\left(\frac{k}{\alpha}+\lambda+\mathrm{i}S\right)\right]_{a_{2+}} \tag{16c}$$

$$b_{2+}=\mathrm{i}C_2\exp\left[\frac{\pi}{2}\left(-\frac{k}{\alpha}+\lambda+\mathrm{i}S\right)\right]_2 - S+(2\mathrm{i}k/\alpha) \qquad a_{2-}=-\exp\left[\pi\left(\frac{k}{\alpha}-\lambda-\mathrm{i}S\right)\right]_{b_{2+}} \tag{16d}$$

where

$$\begin{aligned} C_1&=\frac{\Gamma(\mathrm{i}\lambda-S+\frac{1}{2})\Gamma(-2\mathrm{i}k/\alpha)}{\Gamma[-S-(\mathrm{i}k/\alpha)]\Gamma[\mathrm{i}\lambda+\frac{1}{2}-(\mathrm{i}k/\alpha)]}\\ D_1&=\frac{\Gamma(\mathrm{i}\lambda-S+\frac{1}{2})\Gamma(2\mathrm{i}k/\alpha)}{\Gamma[-S+(\mathrm{i}k/\alpha)]\Gamma[\mathrm{i}\lambda+\frac{1}{2}+(\mathrm{i}k/\alpha)]} \end{aligned} \tag{17a}$$

$$\begin{aligned} C_2&=\frac{\Gamma(\frac{3}{2}+S-\mathrm{i}\lambda)\Gamma(-2\mathrm{i}k/\alpha)}{\Gamma[\frac{1}{2}-\mathrm{i}\lambda-(\mathrm{i}k/\alpha)]\Gamma[1+S-(\mathrm{i}k/\alpha)]}\\ D_2&=\frac{\Gamma(\frac{3}{2}+S-\mathrm{i}\lambda)\Gamma(-2\mathrm{i}k/\alpha)}{\Gamma[\frac{1}{2}-\mathrm{i}\lambda+(\mathrm{i}k/\alpha)]\Gamma[1+S+(\mathrm{i}k/\alpha)]}. \end{aligned} \tag{17b}$$

On using (16) and (17) in (15) and after a considerable amount of algebraic simplification we obtain

$$T(k,S,\lambda)=\frac{\Gamma[-S-(\mathrm{i}k/\alpha)]\Gamma[1+S-(\mathrm{i}k/\alpha)]\Gamma[\frac{1}{2}+\mathrm{i}\lambda-(\mathrm{i}k/\alpha)]\Gamma[\frac{1}{2}-\mathrm{i}\lambda-(\mathrm{i}k/\alpha)]}{\Gamma(-\mathrm{i}k/\alpha)\Gamma[1+(\mathrm{i}k/\alpha)]\Gamma^2[\frac{1}{2}-(\mathrm{i}k/\alpha)]} \tag{18a}$$

$$R(k,S,\lambda)=T(k,S,\lambda)(\cos\pi S\sinh\pi\lambda\ \mathrm{sech}\ \pi k/\alpha+\mathrm{i}\sin\pi S\cosh\pi\lambda\ \mathrm{cosech}\ \pi k/\alpha). \tag{18b}$$

As a check we indeed verify that $|R|^2+|T|^2=1$. Further, we find that the poles of $T(R)$ on the positive imaginary k axis are indeed at $k/\alpha=\mathrm{i}(s-n)$ and correspond to the correct bound-state energies. Finally, we find that $T(R)$ have resonance poles at $k/\alpha=\pm\lambda-(n+\frac{1}{2})\mathrm{i}$.

Proceeding in the same way we have calculated the scattering amplitude for all known shape-invariant potentials. Our results are summarised in tables 1 and 2, where we have given the superpotential $W(x)$, potential $V_-(x)$ and reflection and transmission coefficients (or scattering matrix for three-dimensional problems). It is worthwhile pointing out that a special case of the symmetric Rosen–Morse potential is the δ-function potential in the limit $\alpha\to\infty$ with A held fixed. In that case T and R of table 1 reduce to $T=\mathrm{i}k/(\mathrm{i}k+A)$, $R=A/(\mathrm{i}k+A)$ (Flugge 1971). Similarly the limit $\alpha\to\infty$ for the asymmetric Rosen–Morse potential gives well known results for R and T corresponding to a step potential.

Before finishing this letter we would like to shed some light on the elegant but restricive approach of calculating T and R as given by (9)–(11). In particular, we show that T and R for the potentials of (1) can be obtained in this approach only by exploiting the symmetry of the potential. Since $W_\pm=\pm S\alpha$, (10) and (11) can be written in the form

$$T'\equiv\frac{T(k,S,\lambda)}{\Gamma[S-(\mathrm{i}k/\alpha)+1]\Gamma[-S-(\mathrm{i}k/\alpha)]}=\frac{T(k,S-n,\lambda)}{\Gamma[S-n-(\mathrm{i}k/\alpha)+1]\Gamma[-S+n-(\mathrm{i}k/\alpha)]} \tag{19a}$$

$$R' \equiv \frac{R(k, S, \lambda)}{\Gamma[S-(ik/\alpha)+1]\Gamma[-S-(ik/\alpha)]} = (-1)^n \frac{R(k, S-n, \lambda)}{\Gamma[S-n-(ik/\alpha)+1]\Gamma[-S+n-(ik/\alpha)]}. \tag{19b}$$

From here it is clear that the most general form for T and R is given by

$$\frac{T(k, S, \lambda)}{\Gamma[S-(ik/\alpha)+1]\Gamma[-S-(ik/\alpha)]} = f^{(0)}(k, \lambda) + \sum f_n^{(1)}(k, \lambda) \sin 2n\pi S + \sum f_n^{(2)}(k, \lambda) \cos 2n\pi S \tag{19c}$$

$$\frac{R(k, S, \lambda)}{\Gamma[S-(ik/\alpha)+1]\Gamma[-S-(ik/\alpha)]} = \sum g_n^{(1)}(k, \lambda) \sin(2n+1)\pi S + \sum g_n^{(2)}(k, \lambda) \cos(2n+1)\pi S \tag{19d}$$

where $n = \pm 1, \pm 2, \pm 3, \ldots$. Note that the $g^{(0)}(k, \lambda)$ term in $R'(k, S, \lambda)$ is absent since $R'(k, S, \lambda) = -R'(k, S-1, \lambda)$ while $g(k, \lambda)$ is independent of S! Similar expressions can also be written down for other shape-invariant potentials but it is hard to proceed further and obtain $f^{(n)}$ and $g^{(n)}$ in most cases without having recourse to perturbation theory.

We now make use of the fact that the potentials of (1) (apart from the constant A^2) are invariant under the transformation ($A = S\alpha$, $B = \lambda\alpha$)

$$S \to S' = -\tfrac{1}{2} - i\lambda \qquad \text{and} \qquad \lambda \to \lambda' = i(\tfrac{1}{2} + S) \tag{20}$$

so that

$$\begin{Bmatrix} T \\ R \end{Bmatrix}(k, S, \lambda) = \begin{Bmatrix} T \\ R \end{Bmatrix}(k, -\tfrac{1}{2} - i\lambda, \tfrac{1}{2}i + iS). \tag{21}$$

This condition uniquely fixes the λ dependence of $f^{(n)}$ and $g^{(n)}$ and we obtain

$$\begin{aligned} &\frac{T(k, S, \lambda)}{\Gamma[S+1-(ik/\alpha)]\Gamma[-S-(ik/\alpha)]\Gamma[\frac{1}{2}-(ik/\alpha)-i\lambda]\Gamma[\frac{1}{2}-(ik/\alpha)+i\lambda]} \\ &\quad = f^{(0)}(k) + \sum f_n^{(2)}(k) \cosh 2\pi n\lambda \cos 2\pi S \\ &\qquad + \sum f_n^{(1)}(k) \sinh 2\pi n\lambda \sin 2\pi S \end{aligned} \tag{22a}$$

$$\begin{aligned} &\frac{R(k, S, \lambda)}{\Gamma[S+1-(ik/\alpha)]\Gamma[-S-(ik/\alpha)]\Gamma[\frac{1}{2}-(ik/\alpha)-i\lambda]\Gamma[\frac{1}{2}-(ik/\alpha)+i\lambda]} \\ &\quad = \sum g_n^{(1)}(k) \sin(2n+1)\pi S \cosh(2n+1)\pi\lambda \\ &\quad = \sum g_n^{(2)}(k) \cos(2n+1)\pi S \sinh(2n+1)\pi\lambda. \end{aligned} \tag{22b}$$

Now we also know that for $\lambda = 0$, the potential reduces to symmetric Rosen-Morse potential for which T and R are well known, i.e.

$$\begin{aligned} T(k, S, 0) &= \frac{\Gamma[S+1-(ik/\alpha)]\Gamma[-S-(ik/\alpha)]}{\Gamma(-ik/\alpha)\Gamma[1-(ik/\alpha)]} \\ R(k, S, 0) &= T(k, S, 0) \frac{i \sin \pi S}{\sinh \pi k/\alpha}. \end{aligned} \tag{23}$$

Hence we have from (22) and (23)

$$f_n^{(2)}=0 \qquad f^{(0)}(k)=\frac{1}{\Gamma^2[\frac{1}{2}-(\mathrm{i}k/\alpha)]\Gamma(-\mathrm{i}k/\alpha)\Gamma[1-(\mathrm{i}k/\alpha)]} \tag{24a}$$

$$g_n^{(2)}=\delta_{n1}\frac{\mathrm{i}}{\sinh(\pi k/\alpha)\Gamma^2[\frac{1}{2}-(\mathrm{i}k/\alpha)]\Gamma(-\mathrm{i}k/\alpha)\Gamma[1-(\mathrm{i}k/\alpha)]}. \tag{24b}$$

Finally, notice that as $\lambda\to-\lambda$, the potential (1) merely gets reflected and hence $T(k, S, \lambda)=T(k, S, -\lambda)$. This is turn implies that $f_n^{(1)}=0$ so that one obtains $T(k, S, \lambda)$ as given by (18*a*). Using (18*a*), (22*b*) and (24*b*) and the fact that $|T|^2+|R|^2=1$ one then finds that

$$g_n^{(1)}=\delta_{n1}\frac{1}{\cosh(\pi k/\alpha)\Gamma^2[\frac{1}{2}-(\mathrm{i}k/\alpha)]\Gamma(-\mathrm{i}k/\alpha)\Gamma[1-(\mathrm{i}k/\alpha)]} \tag{25}$$

so that one also obtains the correct expression for $R(k, S, \lambda)$ as given by (18*b*).

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